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Multisymplectic geometry and multisymplectic Preissmann scheme for the KdV equation

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Abstract. The multisymplectic structure of the KdV equation is presented directly from the variational principle. From the numerical view point, we give a multisymplectic twelve-points scheme which is equivalent to the multisymplectic Preissmann scheme. Finally, we test the twelve-points scheme on solitary waves over long time intervals.

(Some figures in this article are in colour only in the electronic version; see www.iop.org)

1. Introduction

Many Hamiltonian PDEs can be described in the language of multisymplectic geometry, for example, the KdV equation, the sine-Gordon equation, the Schrödinger equation, and so on. In addition to allowing one to treat these partial differential equations covariantly, the multisymplectic geometrical method enables one to study the underlying geometrical properties of these partial differential equations. The multisymplectic form plays a very important role in the theory of multisymplectic geometry. The traditional method of giving the multisymplectic form is to construct a Cartan form by the Legendre transformation where the multisymplectic form is the differential of the Cartan form. Recently, for first-order field theory, i.e., the Lagrangian density depends on the state variables and their first-order derivatives, Marsden *et al* [8] presented a method which can obtain the multisymplectic structure in a variational framework completely. However, the Lagrangian density of the KdV equation is not first order, therefore MPS theory cannot be applied directly. In field theory a 'multisymplectic structure' is a manifold with a single higher-order differential form [7–9] also known as the Cartan form. The concept of a multisymplectic geometry in the Bridges' sense is a manifold with a collection of closed two-forms, each of which is non-degenerate on a submanifold [10, 11]. In order to give an invariant framework for the collection of presymplectic two-forms used by Bridges, an interesting approach is to concatenate them into a single three-form which is the differential of the Cartan form in field theory [9]. In this paper, we focus our attention on the KdV equation (whose Lagrangian density is second order) and give the multisymplectic structure of the KdV equation directly from the variational principle.

A basic idea behind the design of numerical schemes is that they can preserve the properties of the original problems as much as possible. From this point of view, the authors of [1–6] have studied some symplectic schemes for computation of Hamiltonian PDEs. Many Hamiltonian

PDEs can be written as multisymplectic equations [10, 11]. Multisymplectic equations have important multisymplectic conservation laws. In the numerical study, we also hope that the numerical approximations can preserve the multisymplectic conservation laws. Bridges and Reich have shown that some numerical discretizations are multisymplectic [10,12,13]. A very useful multisymplectic formula for the KdV equation was given by Bridges [11]. Similar to the method given in [12], we show that the Preissmann scheme is a multisymplectic scheme for the KdV equation. Though the Preissmann scheme is multisymplectic, it involves more computational effort, so we reduce it to a multisymplectic twelve-points scheme. Using the twelve-points scheme, we obtain some numerical results on solitary waves over long time intervals. Compared with the Zabusky–Kruskal scheme [15], the twelve-points scheme gives the same accurate waveforms as the Zabusky–Kruskal scheme, but the twelve-points scheme gives the waveforms for a longer time than the Zabusky–Kruskal scheme.

This paper is organized as follows. In section 2 we describe the multisymplectic geometry of the KdV equation staying entirely in the framework of the variational principle. In section 3 we discuss the multisymplectic Preissmann scheme and reduce it to a multisymplectic twelve-points scheme. Finally, section 4 gives some numerical results on solitary waves over long time intervals.

2. Multisymplectic geometry of the KdV equation

We begin by reviewing a few facts on jet bundles. As an example, we consider the first jet bundle.

The analogue of the configuration space in particle mechanics is the configuration bundle $\pi_{XY}\colon Y \to X$, over an oriented manifold X. Usually, X is the spacetime manifold. We let $T_x X$ denote the tangent space of X at x, and denote the derivative of the map π_{XY} in the direction w by $T_{\pi_{XY}} \cdot w$.

Just as the configuration bundle is the analogue of the configuration space, the jet bundle is the analogue of the tangle bundle.

Definition 2.1. *The first jet bundle over Y is a fibre bundle denoted by $J^1(Y)$ whose fibre over $y \in Y_x = \pi_{XY}^{-1}(x)$, $x \in X$ consists of those linear mappings $\gamma : T_x X \to T_y Y$, satisfying*

$$T_{\pi_{XY}} \circ \gamma = \text{Identity on } T_x X.$$

For simplicity, we let X be the two-dimensional spacetime manifold and the fibre dimension of Y be 1. Coordinates on X are denoted by x, t. The fibre coordinate of Y is denoted by z. We denote the coordinates of the fibre of $J^1(Y)$ by v_x, v_t. We let $\varphi : X \to Y$ be a section of π_{XY}. Corresponding to the section φ, $j^1(\varphi)$ defines a section of $J^1(Y)$. In coordinates, $j^1(\varphi)$ is given by $(x, t, \varphi, \varphi_x, \varphi_t)$, $\varphi_x = \frac{\partial\varphi}{\partial x}$, $\varphi_t = \frac{\partial\varphi}{\partial t}$.

Analogous to the first jet bundle, a higher-order jet bundle $J^m(Y)$ follows as $J^1(J^{m-1}(Y))$ and $j^m(\varphi)$ is a section of $J^m(Y)$.

Above, we only give a brief review of the jet bundle. Some facts on multisymplectic geometry, for example, the dual jet bundle (the analogue of the cotangent), can be found in [7, 8].

The KdV equation can be written as

$$u_t + u_{xxx} + 6uu_x = 0. \tag{2.1}$$

To place the KdV equation in the variational framework, we let $\tau_x = u$. Then, τ satisfies the equation

$$\tau_{xt} + 6\tau_x\tau_{xx} + \tau_{xxxx} = 0. \tag{2.2}$$

The Lagrangian density for equation (2.2) is

$$L(j^2(\varphi)) = (\tfrac{1}{2}\varphi_t\varphi_x + \varphi_x^3 - \tfrac{1}{2}\varphi_{xx}^2)\,\mathrm{d}x \wedge \mathrm{d}t. \tag{2.3}$$

Corresponding to the Lagrangian density $L(j^2(\varphi))$, the action function is defined as follows:

$$S(\varphi) = \int_U Ł(j^2(\varphi)) \qquad U \text{ is an open set of } X. \tag{2.4}$$

Let G be the Lie group of π_{XY} bundle automorphisms η_Y covering η_X. Denote η_Y^λ an smooth path in G such that

$$\bar{\varphi} = \eta_Y^\lambda \circ \varphi \circ (\eta_X^\lambda)^{-1}.$$

The vector field (the infinitesimal generator) of η_Y^λ is

$$V = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \bar{\varphi} = \begin{bmatrix} V^x \\ V^t \\ V^\varphi \end{bmatrix}.$$

We say that φ is a extremum of S if

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} S(\bar{\varphi}) = 0.$$

Now we consider the variation

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} S(\bar{\varphi}) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \int_{\eta_X^\lambda U} \left(\frac{1}{2}\bar{\varphi}_{\bar{t}}\bar{\varphi}_{\bar{x}} + \bar{\varphi}_{\bar{x}}^3 - \frac{1}{2}\bar{\varphi}_{\bar{x}\bar{x}}\right) \mathrm{d}\bar{x} \wedge \mathrm{d}\bar{t}$$

where

$$\eta_X^\lambda \begin{bmatrix} x \\ t \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{t} \end{bmatrix}.$$

A direct computation shows

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} S(\bar{\varphi}) = I_1 + I_2$$

in which

$$I_1 = \int_U (-\varphi_{xt} - 6\varphi_x\varphi_{xx} - \varphi_{xxxx})(V^\varphi - \varphi_t V^t - \varphi_x V^x)\,\mathrm{d}x \wedge \mathrm{d}t$$

$$\begin{aligned} I_2 = \int_{\partial U} & (-\varphi_{xx} V_x^\varphi + \varphi_{xx}\varphi_x V_x^x + \varphi_{xx}\varphi_t V_x^t + \varphi_{xx}\varphi_{xt} V^t)\,\mathrm{d}t + \varphi_{xx}^2 V^t\,\mathrm{d}x - \tfrac{1}{2}\varphi_x(V^\varphi\,\mathrm{d}x \\ & -\varphi_t V^x\,\mathrm{d}t - \varphi_x V^x\,\mathrm{d}x) + (\tfrac{1}{2}\varphi_{xx}^2 - 2\varphi_x^3 - \tfrac{1}{2}\varphi_t\varphi_x - \varphi_x\varphi_{xxx})(V^x\,\mathrm{d}t - V^t\,\mathrm{d}x) \\ & +(\tfrac{1}{2}\varphi_t + 3\varphi_x^2 + \varphi_{xxx})(V^\varphi\,\mathrm{d}t - \varphi_x V^t\,\mathrm{d}x - \varphi_t V^t\,\mathrm{d}t). \end{aligned}$$

By I_2, we can define a Cartan form

$$\begin{aligned} \theta_L = & -\varphi_{xx}\,\mathrm{d}\varphi_x \wedge \mathrm{d}t - \tfrac{1}{2}\varphi_x\,\mathrm{d}\varphi \wedge \mathrm{d}x + (\tfrac{1}{2}\varphi_t + 3\varphi_x^2 + \varphi_{xxx})\,\mathrm{d}\varphi \wedge \mathrm{d}t \\ & +(\tfrac{1}{2}\varphi_{xx}^2 - 2\varphi_x^3 - \tfrac{1}{2}\varphi_t\varphi_x - \varphi_x\varphi_{xxx})\,\mathrm{d}x \wedge \mathrm{d}t. \end{aligned} \tag{2.5}$$

Since

$$\begin{aligned} & j^3(\varphi)^*\,\mathrm{d}x = \mathrm{d}x && j^3(\varphi)^*\,\mathrm{d}t = \mathrm{d}t \\ & j^3(\varphi)^*\,\mathrm{d}\varphi = \varphi_x\,\mathrm{d}x + \varphi_t\,\mathrm{d}t && j^3(\varphi)^*\,\mathrm{d}\varphi_x = \varphi_{xx}\,\mathrm{d}x + \varphi_{xt}\,\mathrm{d}t \end{aligned}$$

we have

$$I_2 = \int_{\partial U} j^3(\varphi)^*(j^3(V) \rfloor \theta_L)$$

where $j^3(V)$ is the jet prolongation of the vector field V [16]. The form θ_L matches the definition of the Cartan form given by Gotay [9] and the multisymplectic form is the 3-form $\Omega_L = -\,\mathrm{d}\theta_L$. Form Ω_L defines a multisymplectic structure on jet bundle $J^3(Y)$.

Above we give the Cartan form θ_L and the multisymplectic form Ω_L, next we consider the Euler–Lagrange equation for the action function $S(\varphi)$.

Noting that $L(j^2(\bar\varphi)) = j^3(\bar\varphi)^*\theta_L$, we have

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0}\int_{\bar U=\eta_X^\lambda U} L(j^2(\bar\varphi)) &= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0}\int_{\bar U} j^3(\bar\varphi)^*\theta_L\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0}\int_{\bar U} j^3(\eta_Y^\lambda\circ\varphi\circ(\eta_X^\lambda)^{-1})^*\theta_L\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0}\int_{\bar U}((\eta_X^\lambda)^{-1})^* j^3(\varphi)^* j^3(\eta_Y^\lambda)^*\theta_L \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0}\int_{U} j^3(\varphi)^* j^3(\eta_Y^\lambda)^*\theta_L\\ &= \int_U j^3(\varphi)^*£_{j^3(V)}\theta_L.\end{aligned} \tag{2.6}$$

Since

$$£_{j^3(V)}\theta_L = -j^3(V)\rfloor\Omega_L + \mathrm{d}(j^3(V)\rfloor\theta_L)$$

where the symbol £ denotes the Lie derivative, we obtain that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} S(\bar\varphi) = -\int_U j^3(\varphi)^*(j^3(V)\rfloor\Omega_L) + \int_{\partial U} j^3(\varphi)^*(j^3(V)\rfloor\theta_L).$$

If V is a vector field with compact support, we have

$$\int_{\partial U} j^3(\varphi)^*(j^3(V)\rfloor\theta_L) = 0.$$

Hence, a necessary condition for φ to be an extremum is that

$$\int_U j^3(\varphi)(j^3(V)\rfloor\Omega_L) = 0 \qquad \text{for any } V \text{ with compact support.} \tag{2.7}$$

We may compute the integrand of (2.6) and obtain that

$$j^3(\varphi)^*(j^3(V)\rfloor\Omega_L) = (\varphi_{xt} + 6\varphi_x\varphi_{xx} + \varphi_{xxxx})(V^\varphi - \varphi_t V^t - \varphi_x V^x). \tag{2.8}$$

Taking the π_{XY}-vertical vector field $V(T_{\pi_{XY}}\cdot V = 0)$ and using the standard method from variational calculus, we obtain that φ satisfies

$$\varphi_{xt} + 6\varphi_x\varphi_{xx} + \varphi_{xxxx} = 0 \tag{2.9}$$

i.e., equation (2.2). So, for any vector field V,

$$j^3(\varphi)^*(j^3(V)\rfloor\Omega_L) = 0 \tag{2.10}$$

holds. A short computation verifies that

$$j^3(\varphi)^*(P\rfloor\Omega_L) = 0 \tag{2.11}$$

where $P \in TJ^3(Y)$ and is $T_{\pi_{Y,J^3(Y)}}$-vertical. For any $W \in TJ^3(Y)$, there exists a vector field V, such that

$$W = j^3(V) + P \tag{2.12}$$

where P is $T_{\pi_{Y,J^3(Y)}}$-vertical. So, by (2.9)–(2.11), if φ is an extremum of S, $j^3(\varphi)^*(W\rfloor\Omega_L)$ must vanish for any vector field $W \in TJ^3(Y)$. Thus, we get the Euler–Lagrange equation

$$j^3(\varphi)^*(W\rfloor\Omega_L) = 0 \tag{2.13}$$

for any vector field $W \in TJ^3(Y)$.

In the following, we consider the multisymplectic form formula and a corollary of the multisymplectic form formula.

Theorem 2.2. *Let η_Y^λ and ξ_Y^λ be two one-parameter symmetry groups of equation (2.12) and the corresponding vector fields be V and W. Then, we have the multisymplectic form formula*

$$\int_{\partial U} j^3(\varphi)^*(j^3(V)\rfloor j^3(W)\rfloor\Omega_L) = 0. \tag{2.14}$$

Proof. Since $j^3[W, V] = [j^3(W), j^3(V)]$, we have

$$\begin{aligned} 0 &= \int_U j^3(\varphi)^*(j^3[W, V]\rfloor\Omega_L) \\ &= \int_U j^3(\varphi)^*([j^3(W), j^3(V)]\rfloor\Omega_L) \\ &= \int_U j^3(\varphi)^*(\pounds_{j^3(W)}(j^3(V)\rfloor\Omega_L) - j^3(V)\rfloor\pounds_{j^3(W)}\Omega_L). \end{aligned} \tag{2.15}$$

η_Y^λ and T_Y^λ are two one-parameter symmetry groups of equation (2.12), so

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} j^3(\eta_Y^\lambda\circ\varphi\circ(\eta_X^\lambda)^{-1})^*(Q\rfloor\Omega_L) &= j^3(\varphi)^*\pounds_{j^3(V)}(Q\rfloor\Omega_L) \\ &= 0 \end{aligned} \tag{2.16}$$

$$\begin{aligned} \left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\right|_{\lambda=0} j^3(\xi_Y^\lambda\circ\varphi\circ(\xi_X^\lambda)^{-1})^*(Q\rfloor\Omega_L) &= j^3(\varphi)^*\pounds_{j^3(W)}(Q\rfloor\Omega_L) \\ &= 0 \end{aligned} \tag{2.17}$$

for any vector field $Q \in TJ^3(Y)$. Thus (2.14) becomes

$$\begin{aligned} 0 &= -\int_U j^3(\varphi)^*(j^3(V)\rfloor\pounds_{j^3(W)}\Omega_L) \\ &= -\int_U j^3(\varphi)^*(j^3(V)\rfloor\,\mathrm{d}(j^3(W)\rfloor\Omega_L)) \\ &= \int_U j^3(\varphi)^*(j^3(V)\rfloor\,\mathrm{d}(\pounds_{j^3(W)}\theta_L)). \end{aligned} \tag{2.18}$$

$J^3(V)\rfloor j^3(W)\rfloor\Omega_L$ can be written as

$$\begin{aligned} J^3(V)\rfloor j^3(W)\rfloor\Omega_L &= j^3(V)\rfloor\,\mathrm{d}(j^3(W)\rfloor\theta_L) - j^3(V)\rfloor\pounds_{j^3(W)}\theta_L \\ &= \pounds_{j^3(V)}(j^3(W)\rfloor\theta_L) - j^3(V)\rfloor\pounds_{j^3(W)}\theta_L - \mathrm{d}(j^3(V)\rfloor j^3(W)\rfloor\theta_L) \end{aligned}$$

so we obtain that

$$\begin{aligned} &\int_{\partial U} j^3(\varphi)^*(j^3(V)\rfloor j^3(W)\rfloor\Omega_L) \\ &\quad= \int_{\partial U} j^3(\varphi)^*(\pounds_{j^3(V)}(j^3(W)\rfloor\theta_L) - j^3(V)\rfloor\pounds_{j^3(W)}\theta_L - \mathrm{d}(j^3(V)\rfloor j^3(W)\rfloor\theta_L)) \\ &\quad= \int_U j^3(\varphi)^*\,\mathrm{d}(\pounds_{j^3(V)}(j^3(W)\rfloor\theta_L) - j^3(V)\rfloor\pounds_{j^3(W)}\theta_L). \end{aligned} \tag{2.19}$$

In addition, since

$$\begin{aligned}\pounds_{j^3(V)}\pounds_{j^3(W)}\theta_L &= j^3(V)\rfloor\,\mathrm{d}(\pounds_{j^3(W)}\theta_L) + \mathrm{d}(j^3(V)\rfloor\pounds_{j^3(W)}\theta_L)\\ &= \pounds_{j^3(V)}(j^3(W)\rfloor(-\Omega_L)) + \pounds_{j^3(V)}\,\mathrm{d}(j^3(W)\rfloor\theta_L)\\ &= -\pounds_{j^3(V)}(j^3(W)\rfloor\Omega_L) + \mathrm{d}\pounds_{j^3(V)}(j^3(W)\rfloor\theta_L)\end{aligned}$$

we have

$$\begin{aligned}&\int_{\partial U} j^3(\varphi)^*(j^3(V)\rfloor j^3(W)\rfloor\Omega_L)\\ &\qquad = \int_U j^3(\varphi)^*(j^3(V)\rfloor\,\mathrm{d}(\pounds_{j^3(W)}\theta_L) + \pounds_{j^3(V)}(j^3(W)\rfloor\Omega_L)).\end{aligned} \tag{2.20}$$

Hence, by (2.15) and (2.17), we get

$$\int_{\partial U} j^3(\varphi)^*(j^3(V)\rfloor j^3(W)\rfloor\Omega_L) = 0.$$

□

In fact, a direct computation verifies that $\mathrm{d}\,\mathrm{d}S = \int_{\partial U} j^3(\varphi)^*(j^3(V)\rfloor j^3(W)\rfloor\Omega_L) = 0$.

Remark. In general, V and W take the form $\xi(x,t,\varphi)\frac{\partial}{\partial x} + \eta(x,t,\varphi)\frac{\partial}{\partial t} + \psi(x,t,\varphi)\frac{\partial}{\partial\varphi}$. If V and W are generalized vector fields (see [16]), i.e., ξ, η, ψ not only depend on x, t, ψ but also on derivatives of φ, a trival modification of the proof of theorem (2.2) ($J^3(Y)$ to $J^\infty(Y)$) shows that theorem (2.2) is also correct. Furthermore, if V and W are infinitesimal symmetries [16] of equation (2.12), theorem 2.2 also holds.

Set $u = \varphi_x$, $v = u_x$, $w = \frac{1}{2}\varphi_t + v_x + V'(u)$, $V(u) = u^3$, then the KdV equation can be reformulated to

$$Mz_t + Kz_x = \nabla_z S(z) \tag{2.21}$$

in which

$$M = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0\\ -\frac{1}{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \qquad K = \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{bmatrix} \qquad z = \begin{bmatrix}\varphi\\ u\\ v\\ w\end{bmatrix}$$

and $S(z) = \frac{1}{2}v^2 - uw + V(u)$. For equation (2.20), there is a conservation law

$$\partial_t(\mathrm{d}z \wedge M\,\mathrm{d}z) + \partial_x(\mathrm{d}z \wedge K\,\mathrm{d}z) = 0. \tag{2.22}$$

Substituting K, M into (2.21) leads to

$$\partial_t(\mathrm{d}\varphi \wedge \mathrm{d}u) + 2\partial_x(\mathrm{d}\varphi \wedge \mathrm{d}w + \mathrm{d}v \wedge \mathrm{d}u) = 0. \tag{2.23}$$

Form $\mathrm{d}\varphi \wedge \mathrm{d}u$ defines a symplectic structure about the time direction and $\mathrm{d}\varphi \wedge \mathrm{d}w$, $\mathrm{d}v \wedge \mathrm{d}u$ define two symplectic structures about the space direction. In this sense, we call conservation law (2.22) a multisymplectic conservation law [10]. In fact, we can view conservation law (2.22) as a corollary of theorem (2.2). Let V, W be π_{XY}-vertical and have the expressions $V^\varphi\frac{\partial}{\partial\varphi}$, $W^\varphi\frac{\partial}{\partial\varphi}$. Then, the corresponding $j^3(V)$ and $j^3(W)$ have the coordinate expressions $(V^\varphi, V^\varphi_x, V^\varphi_t, V^\varphi_{xt}, V^\varphi_{tt}, V^\varphi_{xx}, V^\varphi_{ttx}, V^\varphi_{ttt}, V^\varphi_{xxx}, V^\varphi_{xxt})$ and $(W^\varphi, W^\varphi_x, W^\varphi_t, W^\varphi_{xt}, W^\varphi_{tt}, W^\varphi_{xx}, W^\varphi_{ttx}, W^\varphi_{ttt}, W^\varphi_{xxx}, W^\varphi_{xxt})$. We compute

$$\begin{aligned}j^3(\varphi)^*(j^3(V)\rfloor j^3(W)\rfloor\Omega_L) &= \tfrac{1}{2}(W^\varphi_x V^\varphi - V^\varphi_x W^\varphi)\,\mathrm{d}x + (W^\varphi_{xx}V^\varphi_x - V^\varphi_{xx}W^\varphi_x + \tfrac{1}{2}W^\varphi V^\varphi_t\\ &\quad -\tfrac{1}{2}V^\varphi W^\varphi_t + 6\varphi_x W^\varphi V^\varphi_x - 6\varphi_x V^\varphi W^\varphi_x + W^\varphi V^\varphi_{xxx} - V^\varphi W^\varphi_{xxx})\,\mathrm{d}t.\end{aligned} \tag{2.24}$$

So by Stokes' theorem, we have

$$\int_U -\frac{1}{2}\frac{\partial}{\partial t}(W_x^\varphi V^\varphi - V_x^\varphi W^\varphi)\,\mathrm{d}x\wedge\mathrm{d}t + \frac{\partial}{\partial x}\bigg(W_{xx}^\varphi V_x^\varphi - V_{xx}^\varphi W_x^\varphi + \frac{1}{2}W^\varphi V_t^\varphi - \frac{1}{2}V^\varphi W_t^\varphi + 6\varphi_x W^\varphi V_x^\varphi - 6\varphi_x V^\varphi W_x^\varphi + W^\varphi V_{xxx}^\varphi - V^\varphi W_{xxx}^\varphi\bigg)\,\mathrm{d}x\wedge\mathrm{d}t = 0. \tag{2.25}$$

Since U is arbitary, we obtain

$$\left[\frac{\partial}{\partial t}(W_x^\varphi V^\varphi - V_x^\varphi W^\varphi) - 2\frac{\partial}{\partial x}\bigg(W_{xx}^\varphi V_x^\varphi - V_{xx}^\varphi W_x^\varphi + \frac{1}{2}W^\varphi V_t^\varphi - \frac{1}{2}V^\varphi W_t^\varphi + 6\varphi_x W^\varphi V_x^\varphi - 6\varphi_x V^\varphi W_x^\varphi + W^\varphi V_{xxx}^\varphi - V^\varphi W_{xxx}^\varphi\bigg)\right]\mathrm{d}x\wedge\mathrm{d}t = 0. \tag{2.26}$$

We let $V^u = V_x^\varphi$, $V^v = V_x^u$, $V^w = V_x^v + \frac{1}{2}V_t^\varphi + V''(u)V^u$ and $W^u = W_x^\varphi$, $W^v = W_x^u$, $W^w = W_x^v + \frac{1}{2}W_t^\varphi + W''(u)W^u$. V^φ and W^φ are infinitesimal symmetries of equation (2.12), so $A = (V^\varphi, V^u, V^v, V^w)$ and $B = (W^\varphi, W^u, W^v, W^w)$ are solutions of the variational equations associated with (2.20). Thus, conservation law (2.25) can be reformulated to

$$\partial_t(\mathrm{d}\varphi\wedge\mathrm{d}u)(A, B) + 2\partial_x(\mathrm{d}\varphi\wedge\mathrm{d}w + \mathrm{d}v\wedge\mathrm{d}u)(A, B) = 0$$

which are the conservation of symplecticity given by Bridges [13].

In the numerical study, the multisymplectic conservation law can be used to design multisymplectic schemes, i.e., numerical schemes which can preserve the multisymplectic conservation law.

3. Multisymplectic Preissmann scheme for the KdV equation

In this section, we consider the multisymplectic Preissmann scheme for the KdV equation. Equation (2.20) can be reformulated as

$$u_x = v \tag{3.1}$$

$$v_x = w - \tfrac{1}{2}\varphi_t - V'(u) \tag{3.2}$$

$$w_x = -\tfrac{1}{2}u_t \tag{3.3}$$

$$\varphi_x = u. \tag{3.4}$$

We apply the implicit midpoint scheme to (3.1)–(3.4) and obtain that

$$U_{\frac{1}{2}} = u_k + \frac{\triangle x}{2}V_{\frac{1}{2}} \tag{3.5}$$

$$V_{\frac{1}{2}} = v_k + \frac{\triangle x}{2}\left(W_{\frac{1}{2}} - \frac{1}{2}\partial_t\Phi_{\frac{1}{2}} - V'(U_{\frac{1}{2}})\right) \tag{3.6}$$

$$W_{\frac{1}{2}} = w_k - \frac{\triangle x}{4}\partial_t U_{\frac{1}{2}} \tag{3.7}$$

$$\Phi_{\frac{1}{2}} = \varphi_k + \frac{\triangle x}{2}U_{\frac{1}{2}} \tag{3.8}$$

where $U_{\frac{1}{2}} \approx u(x_k + \frac{\triangle x}{2}, t)$, $V_{\frac{1}{2}} \approx v(x_k + \frac{\triangle x}{2}, t)$, $W_{\frac{1}{2}} \approx w(x_k + \frac{\triangle x}{2}, t)$, $\Phi_{\frac{1}{2}} \approx \varphi(x_k + \frac{\triangle x}{2}, t)$ and $u_k \approx u(x_k, t)$, $v_k \approx v(x_k, t)$, $w_k \approx w(x_k, t)$, $\varphi_k \approx \varphi(x_k, t)$. The u_{k+1}, v_{k+1}, w_{k+1} φ_{k+1} are given by

$$u_{k+1} = u_k + \triangle x V_{\frac{1}{2}} \tag{3.9}$$

$$v_{k+1} = v_k + \triangle x(W_{\frac{1}{2}} - \tfrac{1}{2}\partial_t\Phi_{\frac{1}{2}} - V'(U_{\frac{1}{2}})) \tag{3.10}$$

$$w_{k+1} = w_k - \frac{\triangle x}{2}\partial_t U_{\frac{1}{2}} \tag{3.11}$$

$$\varphi_{k+1} = \varphi_k + \triangle x U_{\frac{1}{2}}. \tag{3.12}$$

For simplicity, we assume $x_k = 0$ and $k = 0$.

We also consider the discretizations of equations (3.2) and (3.3) in time and obtain that

$$U_{i,\frac{1}{2}} = u_{i,0} - \triangle t \partial_x W_{i,\frac{1}{2}} \tag{3.13}$$

$$\Phi_{i,\frac{1}{2}} = \varphi_{i,0} + \frac{\triangle t}{2}\partial_t \Phi_{i,\frac{1}{2}} \tag{3.14}$$

$$u_{i,1} = u_{i,0} - 2\triangle t \partial_x W_{i,\frac{1}{2}} \tag{3.15}$$

$$\varphi_{i,1} = \varphi_{i,0} + \triangle t \partial_t \Phi_{i,\frac{1}{2}} \tag{3.16}$$

here $U_{\mathrm{i},\frac{1}{2}} \approx u(\mathrm{i}\triangle x, \frac{\triangle t}{2})$, $\Phi_{i,\frac{1}{2}} \approx \varphi(\mathrm{i}\triangle x, \frac{\triangle t}{2})$, $u_{i,1} \approx u(\mathrm{i}\triangle x, \triangle t)$, $\varphi_{i,1} \approx \varphi(\mathrm{i}\triangle x, \triangle t)$.

Thus, we get the implicit midpoint discretizations in time and space. In fact, the discretization result leads to the Preissmann scheme†.

By (3.13) and (3.15), we have

$$U_{\frac{1}{2},\frac{1}{2}} = \tfrac{1}{2}(u_{\frac{1}{2},1} + u_{\frac{1}{2},0}). \tag{3.17}$$

Using equations (3.5) and (3.9), we obtain that

$$u_{\frac{1}{2},1} = \tfrac{1}{2}(u_{0,1} + u_{1,1}) \tag{3.18}$$

$$u_{\frac{1}{2},0} = \tfrac{1}{2}(u_{0,0} + u_{1,0}) \tag{3.19}$$

so, (3.17)–(3.19) give

$$U_{\frac{1}{2},\frac{1}{2}} = \tfrac{1}{4}(u_{0,1} + u_{1,1} + u_{0,0} + u_{1,0}). \tag{3.20}$$

Similarly, the relations (3.18)–(3.20) also hold for V, W, Φ. Hence, we have

$$Z_{\frac{1}{2},\frac{1}{2}} = \tfrac{1}{4}(z_{0,1} + z_{1,1} + z_{0,0} + z_{1,0}) \tag{3.21}$$

and

$$z_{\frac{1}{2},1} = \tfrac{1}{2}(z_{0,1} + z_{1,1}) \tag{3.22}$$

$$z_{\frac{1}{2},0} = \tfrac{1}{2}(z_{0,0} + z_{1,0}). \tag{3.23}$$

Using the discretizations in time, analogous to (3.18) and (3.19), we have

$$z_{1,\frac{1}{2}} = \tfrac{1}{2}(z_{1,0} + z_{1,1}) \tag{3.24}$$

$$z_{0,\frac{1}{2}} = \tfrac{1}{2}(z_{0,0} + z_{0,1}). \tag{3.25}$$

By (3.18)–(3.25) and noting that

$$\partial_t Z_{\frac{1}{2},\frac{1}{2}} = \frac{1}{\triangle t}(z_{\frac{1}{2},1} - z_{\frac{1}{2},0}) \tag{3.26}$$

$$\partial_x Z_{\frac{1}{2},\frac{1}{2}} = \frac{1}{\triangle x}(z_{1,\frac{1}{2}} - z_{0,\frac{1}{2}}) \tag{3.27}$$

we obtain the Preissmann scheme

$$\frac{1}{\triangle t}M(z_{\frac{1}{2},1} - z_{\frac{1}{2},0}) + \frac{1}{\triangle x}K(z_{1,\frac{1}{2}} - z_{0,\frac{1}{2}}) = \nabla_z S(Z_{\frac{1}{2},\frac{1}{2}}). \tag{3.28}$$

† The Preissmann scheme is a box scheme and was widely used in hydraulics [17].

The Preissmann scheme is a multisymplectic scheme and has the discretized multisymplectic conservation law

$$\frac{\mathrm{d}\varphi_{\frac{1}{2},1}\wedge\mathrm{d}u_{\frac{1}{2},1}-\mathrm{d}\varphi_{\frac{1}{2},0}\wedge\mathrm{d}u_{\frac{1}{2},0}}{\triangle t}+2\left(\frac{\mathrm{d}\varphi_{1,\frac{1}{2}}\wedge\mathrm{d}w_{1,\frac{1}{2}}-\mathrm{d}\varphi_{0,\frac{1}{2}}\wedge\mathrm{d}w_{0,\frac{1}{2}}}{\triangle x}\right.$$
$$\left.+\frac{\mathrm{d}v_{1,\frac{1}{2}}\wedge\mathrm{d}u_{1,\frac{1}{2}}-\mathrm{d}v_{0,\frac{1}{2}}\wedge\mathrm{d}u_{0,\frac{1}{2}}}{\triangle x}\right)=0 \tag{3.29}$$

which approximates to

$$\begin{aligned}\int_{[0,\triangle x]\times[0,\triangle t]}&\partial_t(\mathrm{d}z\wedge M\,\mathrm{d}z)+\partial_x(\mathrm{d}z\wedge K\,\mathrm{d}z)\\ &=\int_0^{\triangle x}[\mathrm{d}\varphi(x,\triangle t)\wedge\mathrm{d}u(x,\triangle t)-\mathrm{d}\varphi(x,0)\wedge\mathrm{d}u(x,0)]\,\mathrm{d}x\\ &\quad+2\int_0^{\triangle t}[\mathrm{d}\varphi(\triangle x,t)\wedge\mathrm{d}w(\triangle x,t)-\mathrm{d}\varphi(0,t)\wedge\mathrm{d}w(0,t)]\,\mathrm{d}t\\ &\quad+2\int_0^{\triangle t}[\mathrm{d}v(\triangle x,t)\wedge\mathrm{d}u(\triangle x,t)-\mathrm{d}v(0,t)\wedge\mathrm{d}u(0,t)]\,\mathrm{d}t\\ &=0.\end{aligned}$$

Using (3.15) and (3.16), we obtain the identity

$$\begin{aligned}\mathrm{d}\varphi_{\frac{1}{2},1}\wedge\mathrm{d}u_{\frac{1}{2},1}&=\mathrm{d}\varphi_{\frac{1}{2},0}\wedge\mathrm{d}u_{\frac{1}{2},0}-2\triangle t\,\mathrm{d}\varphi_{\frac{1}{2},0}\wedge\partial_x\,\mathrm{d}W_{\frac{1}{2},\frac{1}{2}}\\ &\quad+\triangle t\,\partial_t\,\mathrm{d}\Phi_{\frac{1}{2},\frac{1}{2}}\wedge\mathrm{d}u_{\frac{1}{2},0}-2\triangle t^2\partial_t\,\mathrm{d}\Phi_{\frac{1}{2},\frac{1}{2}}\wedge\partial_x\,\mathrm{d}W_{\frac{1}{2},\frac{1}{2}}.\end{aligned} \tag{3.30}$$

By (3.13) and (3.14), we have

$$\mathrm{d}\varphi_{\frac{1}{2},0}=\mathrm{d}\Phi_{\frac{1}{2},\frac{1}{2}}-\frac{\triangle t}{2}\partial_t\,\mathrm{d}\Phi_{\frac{1}{2},\frac{1}{2}} \tag{3.31}$$

$$\mathrm{d}u_{\frac{1}{2},0}=\mathrm{d}U_{\frac{1}{2},\frac{1}{2}}+\triangle t\,\partial_t\,\mathrm{d}W_{\frac{1}{2},\frac{1}{2}}. \tag{3.32}$$

Combining (3.30)–(3.32), and noting that

$$\partial_x\,\mathrm{d}W_{\frac{1}{2},\frac{1}{2}}=-\tfrac{1}{2}\partial_t\,\mathrm{d}U_{\frac{1}{2},\frac{1}{2}}$$

we get

$$\frac{\mathrm{d}\varphi_{\frac{1}{2},1}\wedge\mathrm{d}u_{\frac{1}{2},1}-\mathrm{d}\varphi_{\frac{1}{2},0}\wedge\mathrm{d}u_{\frac{1}{2},0}}{\triangle t}=\partial t(\mathrm{d}\Phi_{\frac{1}{2},\frac{1}{2}}\wedge\mathrm{d}U_{\frac{1}{2},\frac{1}{2}}). \tag{3.33}$$

Similarly,

$$\frac{\mathrm{d}\varphi_{1,\frac{1}{2}}\wedge\mathrm{d}w_{1,\frac{1}{2}}-\mathrm{d}\varphi_{0,\frac{1}{2}}\wedge\mathrm{d}w_{0,\frac{1}{2}}}{\triangle x}=\mathrm{d}U_{\frac{1}{2},\frac{1}{2}}\wedge\mathrm{d}W_{\frac{1}{2},\frac{1}{2}}-\frac{1}{2}\,\mathrm{d}\Phi_{\frac{1}{2},\frac{1}{2}}\wedge\partial_t\,\mathrm{d}U_{\frac{1}{2},\frac{1}{2}} \tag{3.34}$$

$$\frac{\mathrm{d}v_{1,\frac{1}{2}}\wedge\mathrm{d}u_{1,\frac{1}{2}}-\mathrm{d}v_{0,\frac{1}{2}}\wedge\mathrm{d}u_{0,\frac{1}{2}}}{\triangle x}=\mathrm{d}W_{\frac{1}{2},\frac{1}{2}}\wedge\mathrm{d}U_{\frac{1}{2},\frac{1}{2}}-\frac{1}{2}\partial_t\,\mathrm{d}\Phi_{\frac{1}{2},\frac{1}{2}}\wedge\mathrm{d}U_{\frac{1}{2},\frac{1}{2}}. \tag{3.35}$$

Thus, (3.33)–(3.35) yield our desired result (3.29).

Though the Preissmann scheme (3.28) is multisymplectic, it involves more effort to compute the auxiliary variables w, v, φ, so we eliminate w, v, φ by a trival computation and obtain the following multisymplectic twelve-points scheme (figure 1):

$$\widehat{u_t}+\widehat{u_{xxx}}+\widehat{V''u_x}=0. \tag{3.36}$$

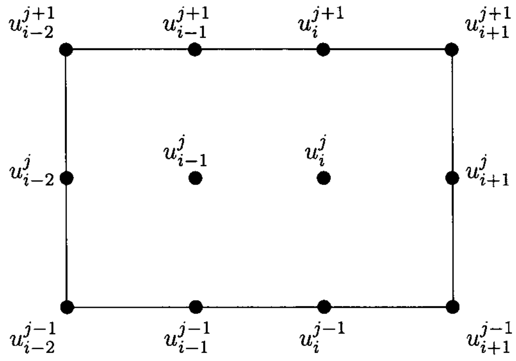

Figure 1. The twelve-points scheme.

If δ_t, δ_x^3, $\bar{u}_i^j$ are defined by

$$\delta_t u_i^j = u_i^{j+1} - u_i^{j-1} \tag{3.37}$$

$$\delta_x^3 u_i^j = u_{i+1}^j - 3u_i^j + 3u_{i-1}^j - u_{i-2}^j \tag{3.38}$$

$$\bar{u}_i^j = \tfrac{1}{4}(u_i^j + u_i^{j+1} + u_{i+1}^j + u_{i+1}^{j+1}) \tag{3.39}$$

then the discretization $\widehat{u_t}$ takes the form

$$\frac{1}{16\triangle t}(\delta_t u_{i+1}^j + 3\delta_t u_i^j + 3\delta_t u_{i-1}^j + \delta_t u_{i-2}^j) \tag{3.40}$$

the discretization $\widehat{u_{xxx}}$ takes the form

$$\frac{1}{4\triangle x^3}(2\delta_x^3 u_i^j + \delta_x^3 u_i^{j+1} + \delta_x^3 u_i^{j-1}) \tag{3.41}$$

and the discretization $\widehat{V'' u_x}$ takes form

$$\frac{1}{4\triangle x}(V'(\bar{u}_i^{j-1}) - V'(\bar{u}_{i-2}^{j-1}) + V'(\bar{u}_i^j) - V'(\bar{u}_{i-2}^j)) \tag{3.42}$$

where $u_m^n \approx u(m\triangle x, n\triangle t)$.

4. Some numerical results on solitary waves

In this section, we test the twelve-points scheme on solitary waves over long time intervals. Concerning the detailed knowledge of solitons for the KdV equation, we refer the reader to [14].

For convenience, we consider the KdV equation

$$u_t + c_1 u_{xxx} + c_2 u u_x = 0 \tag{4.1}$$

here c_1 and c_2 are real constants.

We use the scheme

$$u_i^1 = u_i^0 - \frac{c_2\triangle t}{6\triangle x}(u_{i+1}^0 + u_i^0 + u_{i-1}^0)(u_{i+1}^0 - u_{i-1}^0) - \frac{c_1\triangle t}{2\triangle x^3}(u_{i+2}^0 - 2u_{i+1}^0 + 2u_{i-1}^0 - u_{i-2}^0) \tag{4.2}$$

to give the initial value u_i^1.

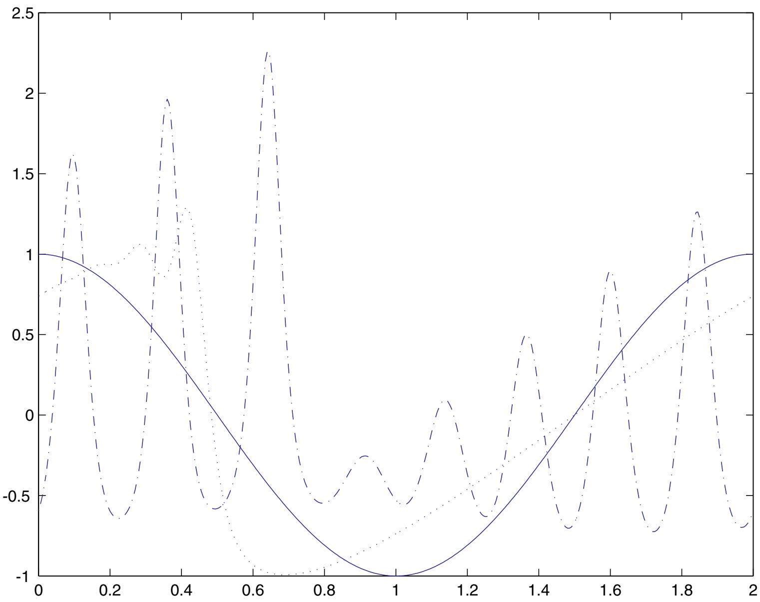

Figure 2. Temporal development of waveform: $\triangle t = 0.005/\pi$, $\triangle x = 2/400$. We denote the waveform for $\pi t = 3.6$ by the full curve, the waveform for $\pi t = 1$ by the dotted curve, and the waveform for $\pi t = 0$ by the dot-dashed curve.,

Choosing $c_1 = 0.022^2$, $c_2 = 1$, we consider equation (4.1) with periodic boundary condition

$$u(0, t) = u(2, t) \tag{4.3}$$

and the initial condition

$$u(x, 0) = \cos(\pi x). \tag{4.4}$$

Figure 2 shows the temporal development of the waveform. The three waveforms are in good agreement with those given by Zabusky and Kruskal in 1965 [15, figure 1].

The numerical scheme used by Zabusky and Kruskal is

$$\begin{aligned} u_i^{j+1} = u_i^{j-1} &- \frac{\Delta t}{3\Delta x}(u_{i+1}^j + u_i^j + u_{i-1}^j)(u_{i+1}^j - u_{i-1}^j) \\ &- \frac{0.022^2 \Delta t}{\Delta x^3}(u_{i+2}^j - 2u_{i+1}^j + 2u_{i-1}^j - u_{i-2}^j). \end{aligned} \tag{4.5}$$

Using scheme (4.5), we performed computations for $\triangle t = \frac{5\times 10^{-n}}{\pi}$, $n = 4, 5$, $\triangle x = \frac{2}{m}$, $m = 200, 300, 400$. From the numerical results, we found that the waveforms given by scheme (4.5) became much worse with the time lapse. In particular, the numerical solutions showed a blow up for $\pi t > 21$. Such a numerical example is shown in figure 3.

We also tested the twelve-points scheme. Figure 4 shows the waveforms given by the twelve-points scheme.

Comparing figures 3 and 4, we find that with increasing time the waveform ($\pi t = 19.9$) given by scheme (4.5) shows a great variation, but the waveform given by the twelve-points scheme is still a smooth one.

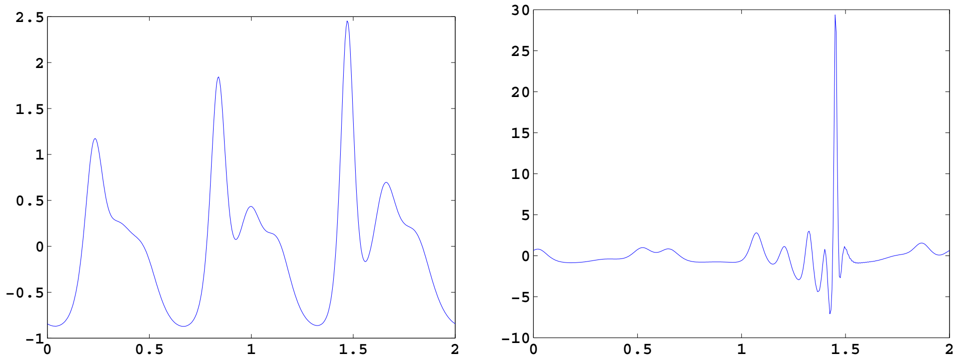

Figure 3. The temporal development of the waveform given by scheme (4.5): $\triangle t = (5 \times 10^{-5})/\pi$, $\triangle x = 2/400$.

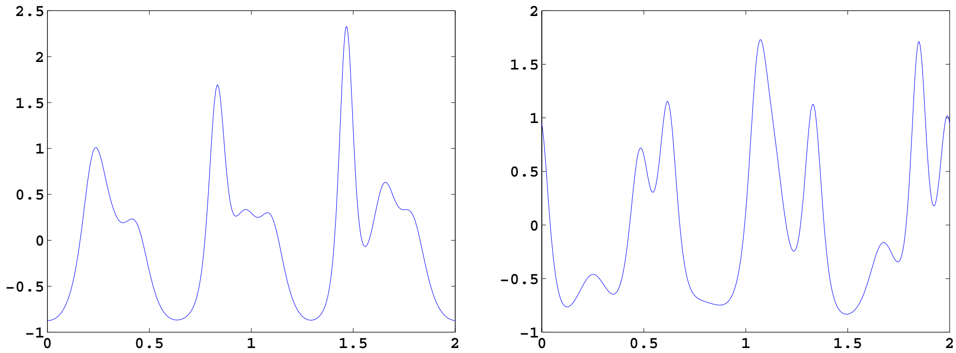

Figure 4. The temporal development of the waveform given by scheme (3.36): $\triangle t = 0.005/\pi$, $\triangle x = 2/400$.

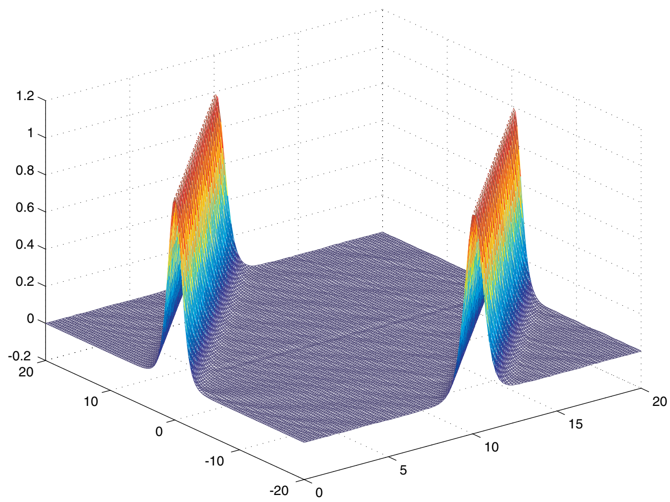

Figure 5. The temporal development of single soliton: $\triangle x = 40/150$, $\triangle t = 0.02$.

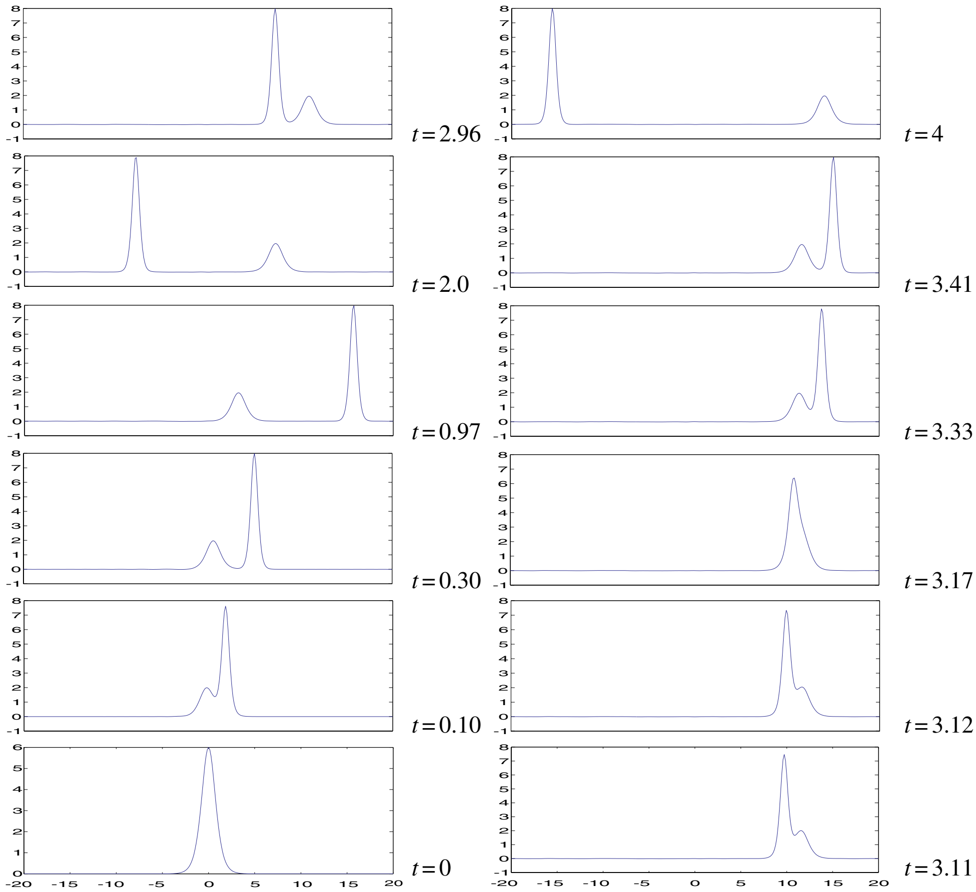

Figure 6. The temporal developments of two solitons: $\triangle x = 40/300$, $\triangle t = 0.002$.

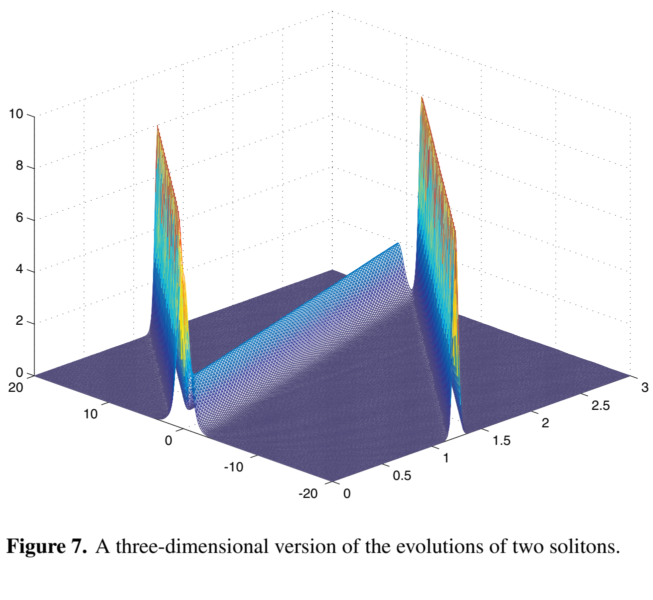

Figure 7. A three-dimensional version of the evolutions of two solitons.

Using the twelve-points scheme, we also give some other numerical results. When $c_1 = 1$, $c_2 = 6$ and the initial condition

$$u(x, 0) = \operatorname{sech}^2\left(\frac{x}{\sqrt{2}}\right) \tag{4.6}$$

equation (4.1) (periodic boundary condition $u(0, t) = u(20, t)$) has one soliton (figure 5). If we take the initial condition

$$u(x, 0) = 6 \operatorname{sech}^2(x) \tag{4.7}$$

equation (4.1) has two solitons (figures 6 and 7).

Figure 6 shows the evolutions of two solitons. At first, the initial profile evolves into two waves and they move apart. The taller wave travels faster, so it catches up and interacts with the shorter one. During the interaction, the taller one passes the shorter one and the two solitary waves all retain their waveforms. Then, they move apart and continue on their way.

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